GEOMETRY – SHEET 1 – Vector Geometry in $\mathbb{R}^n$

1. (i) Show that the distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear (i.e. lie on a line) in $\mathbb{R}^n$ if and only if the vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are linearly dependent.

(ii) Show that the vectors $\mathbf{u}=(1,2,-3)$ and $\mathbf{v}=(6,3,4)$ are perpendicular in $\mathbb{R}^3$. Verify directly Pythagoras' Theorem for the right-angled triangles with vertices $\mathbf{0},\mathbf{u},\mathbf{v}$ and vertices $\mathbf{0},\mathbf{u},\mathbf{u}+\mathbf{v}$.

(iii) Let $\mathbf{v},\mathbf{w}$ be vectors in $\mathbb{R}^n$. Show that if $\mathbf{v}\cdot\mathbf{x}=\mathbf{w}\cdot\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ then $\mathbf{v}=\mathbf{w}$.

2. Consider the two lines in $\mathbb{R}^3$ given parametrically by

$$\mathbf{r}(\lambda)=(1,3,0)+\lambda(2,3,2), \qquad \mathbf{s}(\mu)=(2,1,0)+\mu(0,2,1).$$

Show that the shortest distance between these lines is $\sqrt{3/7}$ by solving the simultaneous equations

$$(\mathbf{r}(\lambda)-\mathbf{s}(\mu))\cdot(2,3,2)=0, \qquad (\mathbf{r}(\lambda)-\mathbf{s}(\mu))\cdot(0,2,1)=0.$$

What geometry do these equations encode? (*Optional* – requires knowledge of partial derivatives. The shortest distance could also be found by solving the equations

$$\frac{\partial}{\partial\lambda}\left(|\mathbf{r}(\lambda)-\mathbf{s}(\mu)|^2\right)=0, \qquad \frac{\partial}{\partial\mu}\left(|\mathbf{r}(\lambda)-\mathbf{s}(\mu)|^2\right)=0.$$

Determine these equations and explain why they are (essentially) the same as the previous two.)

3. Let $(x,y,z)=(s+t+2,3s-2t+1,4s-3t)$. Show that, as s,t vary, the point (x,y,z) ranges over a plane with equation $ax+by+cz=d$ which you should determine.

4. Determine, in the form $\mathbf{r}\cdot\mathbf{n}=c$, the equations of each of the following planes in $\mathbb{R}^3$;
(i) the plane containing the points $(1,0,0)$, $(1,1,0)$, $(0,1,1)$;
(ii) the plane containing the point $(2,1,0)$ and the line $x=y=z$;
(iii) the two planes containing the points $(1,0,1)$, $(0,1,1)$ and which are tangential to the unit sphere, centre $\mathbf{0}$.

5. Given a vector $\mathbf{a}\in\mathbb{R}^2$ and a constant $0<\lambda<1$, define $\mathbf{b}=\mathbf{a}/\left(1-\lambda^2\right)$ and prove that

$$\frac{|\mathbf{r}-\mathbf{a}|^2-\lambda^2|\mathbf{r}|^2}{1-\lambda^2}=|\mathbf{r}-\mathbf{b}|^2-\lambda^2|\mathbf{b}|^2.$$

Deduce *Apollonius' Theorem* which states that if O and A are fixed points in the plane, then the locus of all points X, such that $|AX|=\lambda\,|OX|$, is a circle. Find its centre and radius.

6. *(Optional)* A tetrahedron $ABCD$ has vertices with respective position vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ from an origin O inside the tetrahedron. The lines AO,BO,CO,DO meet the opposite faces in E,F,G,H.

(i) Show that a point lies in the plane BCD if and only if it has position vector $\lambda\mathbf{b}+\mu\mathbf{c}+\nu\mathbf{d}$ where $\lambda+\mu+\nu=1$.

(ii) There are $\alpha,\beta,\gamma,\delta$, not all zero, such that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=\mathbf{0}$. Show that E has position vector

$$\frac{-\alpha\mathbf{a}}{\beta+\gamma+\delta}.$$

(iii) Deduce that

$$\frac{|AO|}{|AE|}+\frac{|BO|}{|BF|}+\frac{|CO|}{|CG|}+\frac{|DO|}{|DH|}=3.$$